

THREE TERM RELATIONS FOR BASIC HYPERGEOMETRIC SERIES

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Abstract. Any three basic hypergeometric series ${}_2\phi_1$ whose respective parameters (a, b, c) differ by integer powers of the base q satisfy a linear relation with coefficients which are rational functions of a, b, c, q and the variable x . These relations are called three term relations for the basic hypergeometric series ${}_2\phi_1$.

This paper gives explicit expressions for the coefficients of these three term relations.

1. Introduction

In this paper, we give explicit expressions for the coefficients of three term relations for the basic hypergeometric series ${}_2\phi_1$, which are q -analogues of Ebisu's [4] results for Gauss hypergeometric series.

The basic hypergeometric series ${}_{p+1}\phi_p$ is defined by

$${}_{p+1}\phi_p \left(\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix} ; q, x \right) = {}_{p+1}\phi_p(a_0, a_1, \dots, a_p; b_1, \dots, b_p; q, x) \\ := \sum_{n=0}^{\infty} \frac{(a_0; q)_n (a_1; q)_n \cdots (a_p; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_p; q)_n} x^n,$$

where q is a fixed complex number satisfying $|q| < 1$, and $(a; q)_n$ denotes the q -shifted factorial defined by $(a; q)_n := (a; q)_\infty / (aq^n; q)_\infty$, $(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n)$. It is assumed that b_1, \dots, b_p are not 1 or negative integer powers of q . For convenience of notation, we denote $(a; q)_n$, $(a; q)_\infty$ and ${}_2\phi_1$ by $(a)_n$, $(a)_\infty$ and ϕ , respectively. Also, we set $a = q^\alpha$, $b = q^\beta$ and $c = q^\gamma$.

The series $\phi(a, b; c; q, x)$ is the q -analogue of Gauss hypergeometric series $F(\alpha, \beta; \gamma; x)$. It is known that for any triples of integers (k, l, m) and (k', l', m') , three hypergeometric series

$$F \left(\begin{matrix} \alpha + k, \beta + l \\ \gamma + m \end{matrix} ; x \right), \quad F \left(\begin{matrix} \alpha + k', \beta + l' \\ \gamma + m' \end{matrix} ; x \right), \quad F \left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix} ; x \right)$$

satisfy a linear relation with coefficients which are rational functions of α, β, γ and x . This relation is called the three term relation for F . Gauss obtained three term relations for F for the cases of

$$(k, l, m), (k', l', m') \in \{(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1), (0, 0, -1)\},$$

where $(k, l, m) \neq (k', l', m')$. That is, there are $\binom{6}{2} = 15$ pairs of (k, l, m) and (k', l', m') . See [11, Chapter 4, p.71] for fifteen relations due to Gauss. Ebisu [4] considered three term

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relations for F for the cases of $(k, l, m) \in \mathbb{Z}^3$ and $(k', l', m') = (1, 1, 1)$:

$$(1.1) \quad F\left(\begin{matrix} \alpha + k, \beta + l \\ \gamma + m \end{matrix}; x\right) = P_1 \cdot F\left(\begin{matrix} \alpha + 1, \beta + 1 \\ \gamma + 1 \end{matrix}; x\right) + P_2 \cdot F\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; x\right),$$

and showed that the coefficients P_1 and P_2 can be expressed as sums of products of two hypergeometric series.

Similarly, for any triples of integers (k, l, m) and (k', l', m') , three basic hypergeometric series

$$\phi\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; q, x\right), \quad \phi\left(\begin{matrix} aq^{k'}, bq^{l'} \\ cq^{m'} \end{matrix}; q, x\right), \quad \phi\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right)$$

satisfy a linear relation with coefficients which are rational functions of a, b, c, q and x . We call this relation the three term relation for ϕ . Fifteen three term relations for ϕ corresponding to Gauss's three term relations for F were obtained by Kim, Rathie and Choi [10].

In this paper, we consider the following three term relation for ϕ corresponding to (1.1).

$$(1.2) \quad \phi\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; q, x\right) = Q \cdot \phi\left(\begin{matrix} aq, bq \\ cq \end{matrix}; q, x\right) + R \cdot \phi\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right).$$

We show that the pair (Q, R) of functions, rational in a, b, c, q and x , are uniquely determined by (k, l, m) , and give explicit expressions for Q and R . It should be noted that if we use these expressions, then we can obtain more general three term relations

$$\phi\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; q, x\right) = Q' \cdot \phi\left(\begin{matrix} aq^{k'}, bq^{l'} \\ cq^{m'} \end{matrix}; q, x\right) + R' \cdot \phi\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right)$$

by eliminating $\phi(aq, bq; cq; x)$ from three term relations (1.2) for (k, l, m) and (k', l', m') .

Throughout this paper, unless otherwise stated explicitly, we assume that

$$E1 : c, \frac{a}{b} \notin q^{\mathbb{Z}}, \quad E2 : a, b, \frac{c}{a}, \frac{c}{b} \notin q^{\mathbb{Z}}.$$

1.1. The coefficients of three term relations for ϕ .

Our main results are the following three theorems. Remark that we only need to consider (1.2) for the cases of $k \leq l$ since $\phi(a, b; c; q, x)$ is symmetric in a and b .

The following theorem says about the uniqueness of the pair (Q, R) satisfying (1.2).

Theorem 1. *For any triple of integers (k, l, m) , the pair (Q, R) of rational functions of a, b, c, q and x , satisfying the three term relation (1.2), are uniquely determined by (k, l, m) .*

The following two theorems give explicit expressions for Q and R . Selectively using these theorems will be helpful when we calculate Q and R (see Remark 5).

Theorem 2. *For any integers k, l and m with $k \leq l$, the coefficients of (1.2) can be expressed as*

$$Q = Q(k, l, m) = -\frac{(1-a)(1-b)c}{(q-c)(1-c)} \frac{x^{1-\max\{m, 0\}}}{(abqx/c)_{\max\{k+l-m, 0\}-1}} P\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right),$$

$$R = R(k, l, m) = -\frac{x^{-\max\{m-1, 0\}}}{(abqx/c)_{\max\{k+l-m-1, 0\}}} P\left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x\right),$$

where P is a polynomial in x defined by

$$P = P\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right) := \begin{cases} \sum_{n=0}^{l-1} (A_n - B_{n-m}) x^n & (m \geq 0, k+l-m \geq 0), \\ \sum_{n=0}^{m-k-1} (\tilde{A}_n - \tilde{B}_{n-m}) x^n & (m \geq 0, k+l-m < 0), \\ \sum_{n=0}^{l-m-1} (A_{n+m} - B_n) x^n & (m < 0, k+l-m \geq 0), \\ \sum_{n=0}^{-k-1} (\tilde{A}_{n+m} - \tilde{B}_n) x^n & (m < 0, k+l-m < 0). \end{cases}$$

Here, A_n , B_n , \tilde{A}_n and \tilde{B}_n are rational functions of a , b , c and q (see Definition 10 for details).

Theorem 3. For any integers k , l and m with $k \leq l$, the coefficients of (1.2) can be expressed as

$$Q = Q(k, l, m) = -\lambda \frac{(1-a)(1-b)c}{(q-c)(1-c)} \frac{x^{1-\max\{m,0\}}}{(abqx/c)_{\max\{k+l-m,0\}-1}} \tilde{P}\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right),$$

$$R = R(k, l, m) = -\lambda' \frac{x^{-\max\{m-1,0\}}}{(abqx/c)_{\max\{k+l-m-1,0\}}} \tilde{P}\left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x\right),$$

where

$$\lambda := (-1)^{k+l-m-1+M} q^{\{k(k-1)+l(l-1)-m(m-1)+M(M-1)\}/2} \frac{(q-c)a^k b^l}{(b-a)c^m} \left(\frac{ab}{c}\right)^M \frac{(c)_m}{(a)_k (b)_l}$$

with $M := \max\{k+l-m, 0\}$,

$$\lambda' := (-1)^{k+l-m+M'} q^{\{k(k-1)+l(l-1)-m(m-1)+M'(M'+1)\}/2} \frac{a^k b^l}{(b-a)c^m} \left(\frac{ab}{c}\right)^{M'-1} \frac{(c)_m}{(aq)_{k-1} (bq)_{l-1}}$$

with $M' := \max\{k+l-m-1, 0\}$ and \tilde{P} is a polynomial in x defined by

$$\tilde{P} = \tilde{P}\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right) := \begin{cases} \sum_{n=0}^{l-1} (C_n - D_{n+k-l}) x^{l-1-n} & (m \geq 0, k+l-m \geq 0), \\ \sum_{n=0}^{m-k-1} (\tilde{C}_n - \tilde{D}_{n+k-l}) x^{m-k-1-n} & (m \geq 0, k+l-m < 0), \\ \sum_{n=0}^{l-m-1} (C_n - D_{n+k-l}) x^{l-m-1-n} & (m < 0, k+l-m \geq 0), \\ \sum_{n=0}^{-k-1} (\tilde{C}_n - \tilde{D}_{n+k-l}) x^{-k-1-n} & (m < 0, k+l-m < 0). \end{cases}$$

Here, C_n , D_n , \tilde{C}_n and \tilde{D}_n are rational functions of a , b , c and q (see Definition 15 for details).

From Theorems 1–3, we obtain the following corollary which gives information about the polynomials P and \tilde{P} .

Corollary 4. *The polynomial P in Theorem 2 and the polynomial \tilde{P} in Theorem 3 satisfy*

$$(1.3) \quad P\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right) = \lambda \cdot \tilde{P}\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right),$$

where λ is defined as in Theorem 3. Therefore, the degree of P equals the degree of \tilde{P} . Let $d := \max\{m, 0\} + \max\{k + l - m, 0\} - k - 1$. Then, for any integers k, l and m with $k \leq l$, the degree of the polynomial \tilde{P} is no more than d , and the coefficient of x^d in \tilde{P} is equal to

$$\begin{cases} \left(\frac{c}{ab}\right)^k q^{k(m-k-l+1)} \{a^{m-2k}(a)_k(aq/c)_{k-m} - b^{m-2k}(b)_k(bq/c)_{k-m}\} & (k = l), \\ \left(\frac{c}{ab}\right)^k q^{k(m-k-l+1)} a^{m-k-l} \frac{(a)_k(aq/c)_{k-m}}{(aq/b)_{k-l}} & (k < l). \end{cases}$$

Remark 5. *Let us call P and \tilde{P} the polynomial parts of Q and R . Then, it follows from (1.3) and the definitions of P and \tilde{P} that Theorem 2 is useful for calculating the coefficients of lower-order terms in the polynomial parts of Q and R , while Theorem 3 is useful for calculating the coefficients of higher-order terms in the polynomial parts.*

From either Theorem 2 or Theorem 3, we obtain the following corollary which gives a relation between Q and R .

Corollary 6. *The coefficients of the three term relation (1.2) satisfy*

$$Q(k-1, l-1, m-1) \Big|_{(a,b,c) \mapsto (aq,bq,cq)} = \frac{(1-aq)(1-bq)x(c-abqx)}{(1-c)(1-cq)} R(k, l, m).$$

2. The coefficients of three term relations for ϕ

In this section, we prove Theorems 1–3 and Corollaries 4, 6. First, we show that Theorem 1. Next, we prove Theorems 2 and 3. For this purpose, we introduce the series

$$f\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) := (1-q)^{\gamma-\alpha-\beta+1} \frac{(q)_\infty(c)_\infty}{(a)_\infty(b)_\infty} \phi\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right),$$

and rewrite the three term relation (1.2) as

$$(2.1) \quad f\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x\right) = \tilde{Q} \cdot f\left(\begin{matrix} aq, bq \\ cq \end{matrix}; x\right) + \tilde{R} \cdot f\left(\begin{matrix} a, b \\ c \end{matrix}; x\right).$$

By comparing (1.2) with (2.1), we find that Q and R can be expressed by \tilde{Q} and \tilde{R} as

$$(2.2) \quad Q = (1-q)^{k+l-m-1} \frac{(cq)_{m-1}}{(aq)_{k-1}(bq)_{l-1}} \tilde{Q},$$

$$(2.3) \quad R = (1-q)^{k+l-m} \frac{(c)_m}{(a)_k(b)_l} \tilde{R}.$$

Thus we investigate \tilde{Q} and \tilde{R} . We define four functions which are local solutions of a q -differential equation $E_q(a, b, c)$ defined by $L_q(a, b, c)y = 0$, where

$$L_q(a, b, c) := x(c-abqx)\Delta_q^2 + \left\{ \frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q} x \right\} \Delta_q - \frac{(1-a)(1-b)}{(1-q)^2}$$

with

$$\Delta_q y(x) = \frac{y(x) - y(qx)}{(1-q)x}.$$

We also define first order q -differential operators called contiguity operators. By using the q -differential equation $E_q(a, b, c)$ and contiguity operators, we obtain linear equations with \tilde{Q} and \tilde{R} , and by solving these equations, we express \tilde{Q} and \tilde{R} as ratios of infinite series.

To express \tilde{Q} and \tilde{R} more explicitly, we introduce Heine's transformation formula for ϕ and Gasper's three formulas for basic hypergeometric series. By using these formulas, we obtain expressions for \tilde{Q} and \tilde{R} as rational functions, and complete the proof of Theorems 2 and 3. Finally, we obtain Corollaries 4 and 6 from Theorems 1–3.

2.1. Proof of Theorem 1.

By using the method due to Vidūnas [12, 3] as reference, we show that Theorem 1. For this purpose, we use the following summation formula which was independently discovered by Bailey [1] and Daum [2] :

$$(2.4) \quad \phi\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q, -\frac{q}{b}\right) = \frac{(-q)_\infty (aq; q^2)_\infty (aq^2/b^2; q^2)_\infty}{(-q/b)_\infty (aq/b)_\infty}, \quad |q| < \min\{1, |b|\}.$$

This is a q -analogue of Kummer's summation formula

$$F\left(\begin{matrix} \alpha, \beta \\ \alpha + 1 - \beta \end{matrix}; -1\right) = \frac{\Gamma(1 + (\alpha/2))\Gamma(\alpha + 1 - \beta)}{\Gamma(1 + \alpha)\Gamma((\alpha/2) + 1 - \beta)}.$$

We also use the three term relation

$$(2.5) \quad \phi(aq, b, cq) = \frac{(1-b)(c-ax)}{c-b} \phi(aq, bq, cq) - \frac{b(1-c)}{c-b} \phi(a, b, c).$$

This follows from Heine's [7, Formulas 24., 43. and 48.] three term relations

$$\begin{aligned} \left(1 - \frac{c}{a}\right) \phi(aq^{-1}, b, c) - \left(1 - \frac{c}{b}\right) \phi(a, bq^{-1}, c) &= \frac{c}{b} \left(1 - \frac{b}{a}\right) \left(1 - \frac{ab}{cq}x\right) \phi(a, b, c), \\ \phi(aq, b, c) - \phi(a, b, c) &= \frac{a(1-b)x}{1-c} \phi(aq, bq, cq), \\ \phi(aq, bq^{-1}, c) - \phi(a, b, c) &= \frac{(aq-b)x}{(1-c)q} \phi(aq, b, cq). \end{aligned}$$

Here, $\phi(a, b, c)$ denote $\phi(a, b; c; q, x)$.

In order to prove by contradiction, let us assume that there are two distinct pairs of rational functions (Q_1, R_1) and (Q_2, R_2) satisfying (1.2). Then, we have

$$(Q_1 - Q_2) \cdot \phi(aq, bq, cq) = (R_2 - R_1) \cdot \phi(a, b, c).$$

This implies that $\phi(aq, bq, cq)/\phi(a, b, c)$ is a rational function of a, b, c, q and x . Therefore, from (2.5),

$$g\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right) := \phi\left(\begin{matrix} aq, b \\ cq \end{matrix}; q, x\right) \Big/ \phi\left(\begin{matrix} a, b \\ c \end{matrix}; q, x\right)$$

is also a rational function of a, b, c, q and x . However, from (2.4), we have

$$\begin{aligned} g\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q, -\frac{q}{b}\right) &= \phi\left(\begin{matrix} aq, b \\ aq^2/b \end{matrix}; q, -\frac{q}{b}\right) \Big/ \phi\left(\begin{matrix} a, b \\ aq/b \end{matrix}; q, -\frac{q}{b}\right) \\ &= \left(1 - \frac{aq}{b}\right) \frac{(aq^2; q^2)_\infty (aq^3/b^2; q^2)_\infty}{(aq; q^2)_\infty (aq^2/b^2; q^2)_\infty}. \end{aligned}$$

It turns out that $g(a) := g(a, b; aq/b; q, -q/b)$ is the rational function of a , while $g(a)$ has unbounded set of poles, so this is a contradiction. Thus Theorem 1 is proved.

2.2. Local solutions of $E_q(a, b, c)$.

We define four functions local solutions of the q -differential equation $E_q(a, b, c)$.

Let y_i ($i = 1, 2, 3, 4$) be the functions defined by

$$\begin{aligned} y_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= y_1(a, b, c; x) := f\left(\begin{matrix} a, b \\ c \end{matrix}; x\right), \\ y_2\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= y_2(a, b, c; x) := x^{1-\gamma} f\left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix}; x\right), \\ y_3\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= y_3(a, b, c; x) := a^{\gamma-\alpha-\beta+1} x^{-\alpha} f\left(\begin{matrix} a, aq/c \\ aq/b \end{matrix}; \frac{cq}{abx}\right), \\ y_4\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) &= y_4(a, b, c; x) := b^{\gamma-\alpha-\beta+1} x^{-\beta} f\left(\begin{matrix} b, bq/c \\ bq/a \end{matrix}; \frac{cq}{abx}\right). \end{aligned}$$

We give two lemmas which are used in Section 2.4 to obtain expressions for \tilde{Q} and \tilde{R} . The following lemma follows from direct calculations.

Lemma 7. *Let $y_i(a, b, c)$ denote $y_i(a, b, c; x)$. Then, we have*

$$\Delta_q y_i(a, b, c) = \begin{cases} y_i(aq, bq, cq) & (i = 1, 2), \\ -\frac{ab}{c} \cdot y_i(aq, bq, cq) & (i = 3, 4). \end{cases}$$

By using a general theory of linear difference equations in [3, Theorem 2.15, p.62], we can verify the following lemma.

Lemma 8. *On the assumption E1, $y_i(a, b, c; x)$ ($i = 1, 2$) are linearly independent solutions around $x = 0$ of $E_q(a, b, c)$, and $y_i(a, b, c; x)$ ($i = 3, 4$) are linearly independent solutions around $x = \infty$.*

2.3. Contiguity operators.

We define first order q -differential operators called contiguity operators, and by combining these operators, we obtain a q -differential operator $H(k, l, m)$ which send the parameters a, b and c to aq^k, bq^l and cq^m , respectively.

Let $H_j(a, b, c)$ and $B_j(a, b, c)$ be the first order q -differential operators defined by

$$\begin{aligned} H_1(a, b, c) &:= \frac{1-a}{1-q} + ax\Delta_q, \\ H_2(a, b, c) &:= \frac{1-b}{1-q} + bx\Delta_q, \\ H_3(a, b, c) &:= \frac{(1-q)c}{(c-a)(c-b)} \left\{ -a - b + ab + \frac{ab}{c} + (1-q)(c-abx)\Delta_q \right\}, \\ B_1(a, b, c) &:= \frac{(1-q)a}{(q-a)(c-a)} \left\{ -q + \frac{cq}{a} + a(1-b)x - (1-q)x(c-abx)\Delta_q \right\}, \\ B_2(a, b, c) &:= \frac{(1-q)b}{(q-b)(c-b)} \left\{ -q + \frac{cq}{b} + b(1-a)x - (1-q)x(c-abx)\Delta_q \right\}, \\ B_3(a, b, c) &:= \frac{q-c}{q(1-q)} + \frac{c}{q}x\Delta_q. \end{aligned}$$

See [7, Formulas 42., 43. and 44., p.292] or [5, Exercises 1.9 (i) and (ii), p.22] for the definitions of H_1, H_2 and B_3 , and see [8, Remark 2.1.4, p.46] for the method for deriving operators B_1, B_2 and H_3 from H_1, H_2 and B_3 , respectively.

The operators H_j ($j = 1, 2, 3$) increase the parameters a , b and c by q times, respectively, while the operators B_j ($j = 1, 2, 3$) decrease these parameters by q times. Hence, we call these six operators contiguity operators. In fact, by performing direct calculations, we can prove the following lemma used in Section 2.4 to obtain expressions for \tilde{Q} and \tilde{R} .

Lemma 9. *Let us denote $y_i(a, b, c; x)$, $y_i(aq^{\pm 1}, b; c; x)$, $y_i(a, bq^{\pm 1}, c; x)$ and $y_i(a, b, cq^{\pm 1}; x)$ by y_i , $y_i(aq^{\pm 1})$, $y_i(bq^{\pm 1})$ and $y_i(cq^{\pm 1})$, respectively. Then, we have*

$$\begin{aligned} H_1(a, b, c)y_i &= y_i(aq), & B_1(a, b, c)y_i &= y_i(aq^{-1}), \\ H_2(a, b, c)y_i &= y_i(bq), & B_2(a, b, c)y_i &= y_i(bq^{-1}), \\ H_3(a, b, c)y_i &= y_i(cq), & B_3(a, b, c)y_i &= y_i(cq^{-1}) \end{aligned}$$

for $i = 1$ and 2 . Also, we have

$$\begin{aligned} H_1(a, b, c)y_i &= -a \cdot y_i(aq), & B_1(a, b, c)y_i &= -\frac{q}{a} \cdot y_i(aq^{-1}), \\ H_2(a, b, c)y_i &= -b \cdot y_i(bq), & B_2(a, b, c)y_i &= -\frac{q}{b} \cdot y_i(bq^{-1}), \\ H_3(a, b, c)y_i &= -\frac{1}{c} \cdot y_i(cq), & B_3(a, b, c)y_i &= -\frac{c}{q} \cdot y_i(cq^{-1}) \end{aligned}$$

for $i = 3$ and 4 .

Let $S_q(a, b, c)$ denote the solution space of $E_q(a, b, c)$ on a simply connected domain in $\mathbb{C} - \{0\}$. From Lemmas 8 and 9, on the assumption E2, the mappings

$$\begin{aligned} H_1(a, b, c) : S_q(a, b, c) &\rightarrow S_q(aq, b, c), \\ H_2(a, b, c) : S_q(a, b, c) &\rightarrow S_q(a, bq, c), \\ H_3(a, b, c) : S_q(a, b, c) &\rightarrow S_q(a, b, cq) \end{aligned}$$

are linear isomorphisms, and their inverse mappings are given by $B_1(aq, b, c)$, $B_2(a, bq, c)$ and $B_3(a, b, cq)$, respectively. Therefore, by combining H_j and B_j ($j = 1, 2, 3$), we obtain a linear isomorphism

$$H(k, l, m) : S_q(a, b, c) \rightarrow S_q(aq^k, bq^l, cq^m)$$

for any integers k , l and m .

2.4. Expressions for \tilde{Q} and \tilde{R} as ratios of infinite series.

We express \tilde{Q} and \tilde{R} as ratios of infinite series defined by the sum of products of the series y_i ($i = 1, 2, 3, 4$).

Let $\mathbb{Q}(a, b, c, q, x)$ denote the field generated over \mathbb{Q} by a , b , c , q and x , and let us denote the ring of polynomials in Δ_q over $\mathbb{Q}(a, b, c, q, x)$ by $\mathbb{Q}(a, b, c, q, x)[\Delta_q]$. By the commutation relation $\Delta_q x = 1 + qx\Delta_q$, we can express $H(k, l, m)$ as

$$(2.6) \quad H(k, l, m) = \rho(\Delta_q) \cdot L_q(a, b, c) + \hat{Q} \cdot \Delta_q + \hat{R},$$

where \hat{Q} , $\hat{R} \in \mathbb{Q}(a, b, c, q, x)$ and $\rho(\Delta_q) \in \mathbb{Q}(a, b, c, q, x)[\Delta_q]$. From Lemmas 7–9 and the definition of y_1 , operating (2.6) to $y_1(a, b, c; x)$ yields

$$(2.7) \quad f\left(\begin{matrix} aq^k, & bq^l \\ & cq^m \end{matrix}; x\right) = \hat{Q} \cdot f\left(\begin{matrix} aq, & bq \\ & cq \end{matrix}; x\right) + \hat{R} \cdot f\left(\begin{matrix} a, & b \\ & c \end{matrix}; x\right).$$

As stated in Section 2.1, the pair (Q, R) in (1.2) are uniquely determined by (k, l, m) . It turns out that the pair (\tilde{Q}, \tilde{R}) in (2.1) are also uniquely determined by (k, l, m) . Therefore, by comparing (2.7) with (2.1), we find that $\hat{Q} = \tilde{Q}$ and $\hat{R} = \tilde{R}$. Namely, we have

$$(2.8) \quad H(k, l, m) = \rho(\Delta_q) \cdot L_q(a, b, c) + \tilde{Q} \cdot \Delta_q + \tilde{R}.$$

From Lemmas 7–9, operating (2.8) to $y_i(a, b; c; x)$ ($i = 1, 2, 3, 4$) yields

$$(2.9) \quad y_i \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) = \tilde{Q} \cdot y_i \left(\begin{matrix} aq, bq \\ cq \end{matrix}; x \right) + \tilde{R} \cdot y_i \left(\begin{matrix} a, b \\ c \end{matrix}; x \right) \quad (i = 1, 2),$$

$$(2.10) \quad \lambda_0 \cdot y_i \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) = -\frac{ab}{c} \tilde{Q} \cdot y_i \left(\begin{matrix} aq, bq \\ cq \end{matrix}; x \right) + \tilde{R} \cdot y_i \left(\begin{matrix} a, b \\ c \end{matrix}; x \right) \quad (i = 3, 4),$$

where $\lambda_0 := (-1)^{k+l-m} a^k b^l c^{-m} q^{\{k(k-1)+l(l-1)-m(m-1)\}/2}$. By solving two equations (2.9) for \tilde{Q} and \tilde{R} , we have

$$(2.11) \quad \tilde{Q} = \tilde{Q}(k, l, m) = \frac{Y \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)}{Y \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)},$$

$$(2.12) \quad \tilde{R} = \tilde{R}(k, l, m) = -\frac{Y \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; x \right)}{Y \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)},$$

where Y is the infinite series defined by

$$(2.13) \quad Y \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) := y_1 \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) y_2 \left(\begin{matrix} a, b \\ c \end{matrix}; x \right) - y_2 \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) y_1 \left(\begin{matrix} a, b \\ c \end{matrix}; x \right).$$

On the other hand, by solving two equations (2.10) for \tilde{Q} and \tilde{R} , we have

$$(2.14) \quad \tilde{Q} = \tilde{Q}(k, l, m) = -\lambda_0 \frac{c}{ab} \frac{\tilde{Y} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)}{\tilde{Y} \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)},$$

$$(2.15) \quad \tilde{R} = \tilde{R}(k, l, m) = -\lambda_0 \frac{\tilde{Y} \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; x \right)}{\tilde{Y} \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right)},$$

where \tilde{Y} is the infinite series defined by

$$(2.16) \quad \tilde{Y} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) := y_3 \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) y_4 \left(\begin{matrix} a, b \\ c \end{matrix}; x \right) - y_4 \left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; x \right) y_3 \left(\begin{matrix} a, b \\ c \end{matrix}; x \right).$$

In the next section, we prove Theorem 2 from (2.11) and (2.12), and we also prove Theorem 3 from (2.14) and (2.15).

2.5. Proofs of Theorems 2 and 3.

We introduce four formulas for basic hypergeometric series. By using these formulas, we obtain expressions for \tilde{Q} and \tilde{R} as rational functions, and prove Theorems 2 and 3.

Heine [7, Formula XVIII., p.325] showed that

$$(\Phi 1) \quad \phi \left(\begin{matrix} a, b \\ c \end{matrix}; q, x \right) = \frac{(abx/c)_\infty}{(x)_\infty} \phi \left(\begin{matrix} c/a, c/b \\ c \end{matrix}; q, \frac{ab}{c} x \right).$$

This is a q -analogue of Euler's transformation formula

$$F\left(\begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; x\right) = (1-x)^{\gamma-\alpha-\beta} F\left(\begin{matrix} \gamma-\alpha, \gamma-\beta \\ \gamma \end{matrix}; x\right).$$

Assume that m_0, m_1, \dots, m_p are non-negative integers. Gasper showed that

$$\begin{aligned} (\Phi 2) \quad & {}_{p+2}\phi_{p+1}\left(\begin{matrix} a_0, b_0, b_1 q^{m_1}, \dots, b_p q^{m_p} \\ b_0 q^{m_0+1}, b_1, \dots, b_p \end{matrix}; q, a_0^{-1} q^{m_0+1-(m_1+\dots+m_p)}\right) \\ &= \frac{(q)_\infty (b_0 q/a_0)_\infty (b_0 q)_{m_0} (b_1/b_0)_{m_1} \cdots (b_p/b_0)_{m_p}}{(b_0 q)_\infty (q/a_0)_\infty (q)_{m_0} (b_1)_{m_1} \cdots (b_p)_{m_p}} b_0^{m_1+\dots+m_p-m_0} \\ &\quad \times {}_{p+2}\phi_{p+1}\left(\begin{matrix} q^{-m_0}, b_0, b_0 q/b_1, \dots, b_0 q/b_p \\ b_0 q/a_0, b_0 q^{1-m_1}/b_1, \dots, b_0 q^{1-m_p}/b_p \end{matrix}; q, q\right), \quad |a_0^{-1} q^{m_0+1-(m_1+\dots+m_p)}| < 1, \\ (\Phi 3) \quad & {}_{p+1}\phi_p\left(\begin{matrix} a_0, b_1 q^{m_1}, \dots, b_p q^{m_p} \\ b_1, \dots, b_p \end{matrix}; q, a_0^{-1} q^{-(m_1+\dots+m_p)}\right) = 0, \quad |a_0^{-1} q^{-(m_1+\dots+m_p)}| < 1, \\ (\Phi 4) \quad & {}_{p+1}\phi_p\left(\begin{matrix} q^{-n}, b_1 q^{m_1}, \dots, b_p q^{m_p} \\ b_1, \dots, b_p \end{matrix}; q, q\right) = 0, \quad n > m_1 + \dots + m_p, \end{aligned}$$

which are (20), (8) and (15) in [6], respectively. The formula $(\Phi 2)$ is a q -analogue of

$$\begin{aligned} & {}_{p+2}F_{p+1}\left(\begin{matrix} \alpha, \beta, \beta_1 + m_1, \dots, \beta_p + m_p \\ \beta + \gamma + 1, \beta_1, \dots, \beta_p \end{matrix}; 1\right) \\ &= \frac{\Gamma(\beta + \gamma + 1)\Gamma(1 - \alpha)}{\Gamma(\beta + 1 - \alpha)\Gamma(\gamma + 1)} \frac{(\beta_1 - \beta, m_1) \cdots (\beta_p - \beta, m_p)}{(\beta_1, m_1) \cdots (\beta_p, m_p)} \\ &\quad \times {}_{p+2}F_{p+1}\left(\begin{matrix} -\gamma, \beta, 1 + \beta - \beta_1, \dots, 1 + \beta - \beta_p \\ \beta + 1 - \alpha, 1 + \beta - \beta_1 - m_1, \dots, 1 + \beta - \beta_p - m_p \end{matrix}; 1\right), \end{aligned}$$

provided $\Re(\gamma - \alpha) > m_1 + \dots + m_p - 1$, where $(\alpha, n) := \Gamma(\alpha + n)/\Gamma(\alpha)$. This formula was also given by Gasper [6, (18)]. Both $(\Phi 3)$ and $(\Phi 4)$ are q -analogues of Karlsson's [9, (12)] summation formula

$${}_{p+1}F_p\left(\begin{matrix} \alpha, \beta_1 + m_1, \dots, \beta_p + m_p \\ \beta_1, \dots, \beta_p \end{matrix}; 1\right) = 0, \quad \Re(-\alpha) > m_1 + \dots + m_p.$$

First, we prove Theorem 2. The following is the definition of P in Theorem 2.

Definition 10. For any integers k, l, m with $k \leq l$, let P be the polynomial in x defined by

$$P = P\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x\right) := \begin{cases} \sum_{n=0}^{l-1} (A_n - B_{n-m}) x^n & (m \geq 0, k + l - m \geq 0), \\ \sum_{n=0}^{m-k-1} (\tilde{A}_n - \tilde{B}_{n-m}) x^n & (m \geq 0, k + l - m < 0), \\ \sum_{n=0}^{l-m-1} (A_{n+m} - B_n) x^n & (m < 0, k + l - m \geq 0), \\ \sum_{n=0}^{-k-1} (\tilde{A}_{n+m} - \tilde{B}_n) x^n & (m < 0, k + l - m < 0), \end{cases}$$

where $A_n = B_n = \tilde{A}_n = \tilde{B}_n := 0$ for any negative integer n , and

$$A_n := \frac{-c(aq/c)_{k-m}(bq/c)_{l-m}}{(q^2/c)_{-m-1}} \frac{(c)_{m-n-1}}{(q^{-n})_n (a)_{k-n} (b)_{l-n}} q^{m-n-1} {}_4\phi_3\left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q\right),$$

$$\begin{aligned}
B_n &:= \frac{(aq/c)_n (bq/c)_n}{(q)_n (q^2/c)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, cq^{-n-1}, cq^{m-k}/a, cq^{m-l}/b \\ cq^m, cq^{-n}/a, cq^{-n}/b \end{matrix} ; q, q^{k+l-m+1} \right), \\
\tilde{A}_n &:= \frac{(c)_m}{(a)_k (b)_l} \frac{(aq/c)_{n+k-m} (bq/c)_{n+l-m}}{(q)_n (q^2/c)_{n-m}} {}_4\phi_3 \left(\begin{matrix} q^{-n}, cq^{m-n-1}, c/a, c/b \\ c, cq^{m-k-n}/a, cq^{m-l-n}/b \end{matrix} ; q, q^{m-k-l+1} \right), \\
\tilde{B}_n &:= \frac{(aq^{-n})_n (bq^{-n})_n}{(q^{-n})_n (cq^{-n-1})_n} q^{-n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, cq^{-n-1}, aq^k, bq^l \\ cq^m, aq^{-n}, bq^{-n} \end{matrix} ; q, q \right)
\end{aligned}$$

for any non-negative integer n .

To prove Theorem 2, we give some lemmas. The following lemma follows from **(Φ1)**.

Lemma 11. *The series Y can be written as*

$$\begin{aligned}
(2.17a) \quad Y \left(\begin{matrix} k, l \\ m \end{matrix} ; \begin{matrix} a, b \\ c \end{matrix} ; x \right) &= -\lambda_1 (1-q)^{m-k-l} x^{1-\gamma-m} \frac{(a)_k (b)_l}{(c)_m} \frac{(abq^{k+l-m}x/c)_\infty}{(x)_\infty} \left\{ \sum_{n=0}^{\infty} A_n x^n - x^m \sum_{n=0}^{\infty} B_n x^n \right\}, \\
(2.17b) \quad &= -\lambda_1 (1-q)^{m-k-l} x^{1-\gamma-m} \frac{(a)_k (b)_l}{(c)_m} \frac{(abx/c)_\infty}{(x)_\infty} \left\{ \sum_{n=0}^{\infty} \tilde{A}_n x^n - x^m \sum_{n=0}^{\infty} \tilde{B}_n x^n \right\},
\end{aligned}$$

where

$$\lambda_1 := (1-q)^{2(\gamma-\alpha-\beta+1)} \frac{(q)_\infty^2 (c)_\infty (q^2/c)_\infty}{(a)_\infty (b)_\infty (aq/c)_\infty (bq/c)_\infty}.$$

Proof. Applying **(Φ1)** to $y_1(aq^k, bq^l; cq^m; x)$ and $y_2(aq^k, bq^l; cq^m; x)$ in **(2.13)** yields

$$\begin{aligned}
Y \left(\begin{matrix} k, l \\ m \end{matrix} ; \begin{matrix} a, b \\ c \end{matrix} ; x \right) &= (1-q)^{2(\gamma-\alpha-\beta+1)+m-k-l} \frac{(q)_\infty^2 (cq^m)_\infty (q^2/c)_\infty}{(aq^k)_\infty (bq^l)_\infty (aq/c)_\infty (bq/c)_\infty} x^{1-\gamma-m} \frac{(abq^{k+l-m}x/c)_\infty}{(x)_\infty} \\
&\times \left\{ x^m \phi \left(\begin{matrix} cq^{m-k}/a, cq^{m-l}/b \\ cq^m \end{matrix} ; q, \frac{abq^{k+l-m}}{c} x \right) \phi \left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix} ; q, x \right) \right. \\
&\quad \left. - \frac{(aq/c)_{k-m} (bq/c)_{l-m} (c)_m}{(q^2/c)_{-m} (a)_k (b)_l} \phi \left(\begin{matrix} q^{1-k}/a, q^{1-l}/b \\ q^{2-m}/c \end{matrix} ; q, \frac{abq^{k+l-m}}{c} x \right) \phi \left(\begin{matrix} a, b \\ c \end{matrix} ; q, x \right) \right\}.
\end{aligned}$$

From $(a)_{n-i} = (-a)^{-i} q^{(1-n)i + \binom{i}{2}} (a)_n / (a^{-1} q^{1-n})_i$, we can find that the coefficient of x^n in

$$\phi \left(\begin{matrix} cq^{m-k}/a, cq^{m-l}/b \\ cq^m \end{matrix} ; q, \frac{abq^{k+l-m}}{c} x \right) \phi \left(\begin{matrix} aq/c, bq/c \\ q^2/c \end{matrix} ; q, x \right)$$

is equal to B_n , and the coefficient of x^n in

$$\frac{(aq/c)_{k-m} (bq/c)_{l-m} (c)_m}{(q^2/c)_{-m} (a)_k (b)_l} \phi \left(\begin{matrix} q^{1-k}/a, q^{1-l}/b \\ q^{2-m}/c \end{matrix} ; q, \frac{abq^{k+l-m}}{c} x \right) \phi \left(\begin{matrix} a, b \\ c \end{matrix} ; q, x \right)$$

is equal to A_n . This proves **(2.17a)**.

Applying $(\Phi 1)$ to $y_1(a, b; c; x)$ and $y_2(a, b; c; x)$ in (2.13) yields

$$\begin{aligned} Y\left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x\right) \\ = (1-q)^{2(\gamma-\alpha-\beta+1)+m-k-l} \frac{(q)_\infty^2 (cq^m)_\infty (q^2/c)_\infty}{(aq^k)_\infty (bq^l)_\infty (aq/c)_\infty (bq/c)_\infty} x^{1-\gamma-m} \frac{(abx/c)_\infty}{(x)_\infty} \\ \times \left\{ x^m \phi\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; q, x\right) \phi\left(\begin{matrix} q/a, q/b \\ q^2/c \end{matrix}; q, \frac{ab}{c}x\right) \right. \\ \left. - \frac{(aq/c)_{k-m} (bq/c)_{l-m} (c)_m}{(q^2/c)_{-m} (a)_k (b)_l} \phi\left(\begin{matrix} aq^{k+1-m}/c, bq^{l+1-m}/c \\ q^2-m/c \end{matrix}; q, x\right) \phi\left(\begin{matrix} c/a, c/b \\ c \end{matrix}; q, \frac{ab}{c}x\right) \right\}. \end{aligned}$$

From $(a)_{n-i} = (-a)^{-i} q^{(1-n)i + \binom{i}{2}} (a)_n / (a^{-1} q^{1-n})_i$, we can find that the coefficient of x^n in

$$\phi\left(\begin{matrix} aq^k, bq^l \\ cq^m \end{matrix}; q, x\right) \phi\left(\begin{matrix} q/a, q/b \\ q^2/c \end{matrix}; q, \frac{ab}{c}x\right)$$

is equal to \tilde{B}_n , and the coefficient of x^n in

$$\frac{(aq/c)_{k-m} (bq/c)_{l-m} (c)_m}{(q^2/c)_{-m} (a)_k (b)_l} \phi\left(\begin{matrix} aq^{k+1-m}/c, bq^{l+1-m}/c \\ q^2-m/c \end{matrix}; q, x\right) \phi\left(\begin{matrix} c/a, c/b \\ c \end{matrix}; q, \frac{ab}{c}x\right)$$

is equal to \tilde{A}_n . This proves $(2.17b)$. \square

From $(\Phi 2)$ – $(\Phi 4)$, we obtain the following lemma.

Lemma 12. *For any integers k, l and m with $k \leq l$, we have*

- (i) $m \geq 0, k + l - m \geq 0 \Rightarrow A_n - B_{n-m} = 0 \quad (n \geq l).$
- (ii) $m \geq 0, k + l - m < 0 \Rightarrow \tilde{A}_n - \tilde{B}_{n-m} = 0 \quad (n \geq m - k).$
- (iii) $m < 0, k + l - m \geq 0 \Rightarrow A_{n+m} - B_n = 0 \quad (n \geq l - m).$
- (iv) $m < 0, k + l - m < 0 \Rightarrow \tilde{A}_{n+m} - \tilde{B}_n = 0 \quad (n \geq -k).$

Proof. Let us prove (i). First, by using $(\Phi 2)$, we show that

$$(i-1) \quad m \geq 0, k + l - m \geq 0 \Rightarrow A_n - B_{n-m} = 0 \quad (n \geq \max\{l, m\}).$$

By substituting $p = 2$, $a_0 = q^{m-n}$, $b_0 = cq^{m-n-1}$, $b_1 = cq^{m-n}/a$, $b_2 = cq^{m-n}/b$, $m_0 = n$, $m_1 = n - k$ and $m_2 = n - l$ into $(\Phi 2)$, we find that if $k + l - m \geq 0$ and $n \geq \max\{0, l\}$, then

$$\begin{aligned} & {}_4\phi_3\left(\begin{matrix} q^{m-n}, cq^{m-n-1}, cq^{m-k}/a, cq^{m-l}/b \\ cq^m, cq^{m-n}/a, cq^{m-n}/b \end{matrix}; q, q^{k+l-m+1}\right) \\ &= \frac{(q)_\infty (c)_\infty (cq^{m-n})_n (q/a)_{n-k} (q/b)_{n-l}}{(cq^{m-n})_\infty (q^{n+1-m})_\infty (q)_n (cq^{m-n}/a)_{n-k} (cq^{m-n}/b)_{n-l}} c^{n-k-l} q^{(m-n-1)(n-k-l)} \\ & \times {}_4\phi_3\left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q\right) \\ &= \frac{(c)_m (q/a)_{n-k} (q/b)_{n-l}}{(q^{n+1-m})_m (cq^{m-n}/a)_{n-k} (cq^{m-n}/b)_{n-l}} c^{n-k-l} q^{(m-n-1)(n-k-l)} {}_4\phi_3\left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q\right). \end{aligned}$$

Thus, it turns out that if $k + l - m \geq 0$ and $n \geq \max\{0, l, m\}$, then

$$\begin{aligned} B_{n-m} &= \frac{(aq/c)_{n-m} (bq/c)_{n-m}}{(q)_{n-m} (q^2/c)_{n-m}} \\ & \times \frac{(c)_m (q/a)_{n-k} (q/b)_{n-l} c^{n-k-l}}{(q^{n+1-m})_m (cq^{m-n}/a)_{n-k} (cq^{m-n}/b)_{n-l}} q^{(m-n-1)(n-k-l)} {}_4\phi_3\left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q\right) \end{aligned}$$

$$\begin{aligned}
&= \frac{-c(aq/c)_{k-m}(bq/c)_{l-m}}{(q^2/c)_{-m-1}} \frac{(c)_{m-n-1}}{(q^{-n})_n(a)_{k-n}(b)_{l-n}} q^{m-n-1} {}_4\phi_3 \left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q \right) \\
&= A_n.
\end{aligned}$$

This proves (i-1). Next, by using $(\Phi 4)$, we show that

$$(i-2) \quad m \geq 0, k+l-m \geq 0, l < m \Rightarrow A_n - B_{n-m} = 0 \quad (l \leq n \leq m-1).$$

By substituting $p = 3$, $b_1 = c$, $b_2 = aq^{k-n}$, $b_3 = bq^{l-n}$, $m_1 = m-n-1$, $m_2 = n-k$ and $m_3 = n-l$ into $(\Phi 4)$, we find that if $k+l-m \geq 0$, $l < m$ and $l \leq n \leq m-1$, then

$${}_4\phi_3 \left(\begin{matrix} q^{-n}, cq^{m-n-1}, a, b \\ c, aq^{k-n}, bq^{l-n} \end{matrix}; q, q \right) = 0.$$

Thus, it turns out that if $k+l-m \geq 0$, $l < m$ and $l \leq n \leq m-1$, then $A_n - B_{n-m} = A_n = 0$. This proves (i-2). Consequently, from (i-1) and (i-2), we have (i).

In the same way as we proved (i-1), by using $(\Phi 2)$, we can show that

$$(ii-1) \quad m \geq 0, k+l-m < 0 \Rightarrow \tilde{A}_n - \tilde{B}_{n-m} = 0 \quad (n \geq \max\{m, m-k\}).$$

$$(iii-1) \quad m < 0, k+l-m \geq 0 \Rightarrow A_{n+m} - B_n = 0 \quad (n \geq \max\{-m, l-m\}).$$

$$(iv-1) \quad m < 0, k+l-m < 0 \Rightarrow \tilde{A}_{n+m} - \tilde{B}_n = 0 \quad (n \geq \max\{-m, -k\}).$$

Also, in the same way as we proved (i-2), by using $(\Phi 3)$ or $(\Phi 4)$, we can show that

$$(ii-2) \quad m \geq 0, k+l-m < 0, m-k < m \Rightarrow \tilde{A}_n - \tilde{B}_{n-m} = 0 \quad (m-k \leq n \leq m-1).$$

$$(iii-2) \quad m < 0, k+l-m \geq 0, l-m < -m \Rightarrow A_{n+m} - B_n = 0 \quad (l-m \leq n \leq -m-1).$$

$$(iv-2) \quad m < 0, k+l-m < 0, -k < -m \Rightarrow \tilde{A}_{n+m} - \tilde{B}_n = 0 \quad (-k \leq n \leq -m-1).$$

From these results, we can complete the proof of Lemma 12. \square

It immediately follows from Lemmas 11 and 12 that

Lemma 13. For any integers k, l and m with $k \leq l$, the series Y can be expressed as

$$\begin{aligned}
(2.18) \quad Y \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) &= -\lambda_1 (1-q)^{m-k-l} \frac{(a)_k (b)_l}{(c)_m} \\
&\quad \times x^{1-\gamma-\max\{m,0\}} \frac{(abq^{\max\{k+l-m,0\}}x/c)_\infty}{(x)_\infty} P \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x \right),
\end{aligned}$$

where λ_1 is defined as in Lemma 11.

From (2.11), (2.12) and Lemma 13, we obtain the following lemma which gives expressions for \tilde{Q} and \tilde{R} as rational functions.

Lemma 14. Assume that $k \leq l$. Then, the coefficients of (2.1) can be expressed as

$$\begin{aligned}
\tilde{Q}(k, l, m) &= -(1-q)^{m-k-l+1} \frac{c(a)_k (b)_l}{(q-c)(c)_m} \frac{x^{1-\max\{m,0\}}}{(abqx/c)_{\max\{k+l-m,0\}-1}} P \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x \right), \\
\tilde{R}(k, l, m) &= -(1-q)^{m-k-l} \frac{(a)_k (b)_l}{(c)_m} \frac{x^{-\max\{m-1,0\}}}{(abqx/c)_{\max\{k+l-m-1,0\}}} P \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x \right).
\end{aligned}$$

Proof. By substituting $(k, l, m) = (1, 1, 1)$ into (2.18), we have

$$(2.19) \quad Y \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \lambda_1 \frac{x^{-\gamma}}{1-q} \frac{(abqx/c)_\infty}{(x)_\infty} \frac{q-c}{c}.$$

From (2.11), (2.18) and (2.19), we have the expression for \tilde{Q} .

Replacing k, l, m, a, b, c in (2.18) with respective $k-1, l-1, m-1, aq, bq, cq$ yields

$$(2.20) \quad Y \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; x \right) \\ = \lambda_1 (1-q)^{m-k-l-1} \frac{q-c}{c} \frac{(a)_k (b)_l}{(c)_m} \\ \times x^{-\gamma - \max\{m-1, 0\}} \frac{(abq^{\max\{k+l-m-1, 0\}+1} x/c)_\infty}{(x)_\infty} P \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x \right).$$

From (2.12), (2.19) and (2.20), we have the expression for \tilde{R} . Thus the lemma is proved. \square

From (2.2), (2.3) and Lemma 14, we can complete the proof of Theorem 2.

Next, we prove Theorem 3. The following is the definition of \tilde{P} in Theorem 3.

Definition 15. For any integers k, l, m with $k \leq l$, let \tilde{P} be the polynomial in x defined by

$$\tilde{P} = \tilde{P} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x \right) := \begin{cases} \sum_{n=0}^{l-1} (C_n - D_{n+k-l}) x^{l-1-n} & (m \geq 0, k+l-m \geq 0), \\ \sum_{n=0}^{m-k-1} (\tilde{C}_n - \tilde{D}_{n+k-l}) x^{m-k-1-n} & (m \geq 0, k+l-m < 0), \\ \sum_{n=0}^{l-m-1} (C_n - D_{n+k-l}) x^{l-m-1-n} & (m < 0, k+l-m \geq 0), \\ \sum_{n=0}^{-k-1} (\tilde{C}_n - \tilde{D}_{n+k-l}) x^{-k-1-n} & (m < 0, k+l-m < 0), \end{cases}$$

where $D_n = \tilde{D}_n := 0$ for any negative integer n , and

$$C_n := \mu_1 \frac{(b)_n (bq/c)_n}{(q)_n (bq/a)_n} \left(\frac{cq}{ab} \right)^n {}_4\phi_3 \left(\begin{matrix} q^{-n}, aq^{-n}/b, q^{1-l}/b, cq^{m-l}/b \\ aq^{k-l+1}/b, q^{1-n}/b, cq^{-n}/b \end{matrix}; q, q \right), \\ D_n := \mu_2 \frac{(q^{1-k}/a)_n (cq^{m-k}/a)_n}{(q)_n (bq^{l-k+1}/a)_n} q^n {}_4\phi_3 \left(\begin{matrix} q^{-n}, aq^{k-l-n}/b, a, aq/c \\ aq/b, aq^{k-n}, aq^{k-m-n+1}/c \end{matrix}; q, q^{k+l-m+1} \right), \\ \tilde{C}_n := \mu_1 \frac{(aq^k)_n (aq^{k-m+1}/c)_n}{(q)_n (aq^{k-l+1}/b)_n} \left(\frac{cq^{m-k-l+1}}{ab} \right)^n {}_4\phi_3 \left(\begin{matrix} q^{-n}, bq^{l-k-n}/a, q/a, c/a \\ bq/a, q^{1-k-n}/a, cq^{m-k-n}/a \end{matrix}; q, q \right), \\ \tilde{D}_n := \mu_2 \frac{(q/b)_n (c/b)_n}{(q)_n (aq/b)_n} q^n {}_4\phi_3 \left(\begin{matrix} q^{-n}, bq^{-n}/a, bq^l, bq^{l-m+1}/c \\ bq^{l-k+1}/a, bq^{-n}, bq^{1-n}/c \end{matrix}; q, q^{m-k-l+1} \right)$$

with

$$\mu_1 := a^{m-k-l} \left(\frac{c}{ab} \right)^k q^{k(m-k-l+1)} \frac{(a)_k (aq/c)_{k-m}}{(aq/b)_{k-l}}, \\ \mu_2 := b^{m-k-l} \left(\frac{c}{ab} \right)^l q^{l(m-k-l+1)} \frac{(b)_l (bq/c)_{l-m}}{(bq/a)_{l-k}}$$

for any non-negative integer n .

To prove Theorem 3, we give some lemmas. In the same way as we proved Lemma 11, by using (Φ1), we can show that

Lemma 16. The series \tilde{Y} can be written as

$$\tilde{Y} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \lambda_2 (1-q)^{m-k-l} x^{-\alpha-\beta-k} \frac{(q/x)_\infty}{(cq^{m-k-l+1}/(abx))_\infty} \left\{ \sum_{n=0}^{\infty} C_n x^{-n} - x^{k-l} \sum_{n=0}^{\infty} D_n x^{-n} \right\},$$

$$= \lambda_2 (1-q)^{m-k-l} x^{-\alpha-\beta-k} \frac{(q/x)_\infty}{(cq/(abx))_\infty} \left\{ \sum_{n=0}^{\infty} \tilde{C}_n x^{-n} - x^{k-l} \sum_{n=0}^{\infty} \tilde{D}_n x^{-n} \right\},$$

where

$$\lambda_2 := (1-q)^{2(\gamma-\alpha-\beta+1)} (ab)^{\gamma-\alpha-\beta+1} \frac{(q)_\infty^2 (aq/b)_\infty (bq/a)_\infty}{(a)_\infty (b)_\infty (aq/c)_\infty (bq/c)_\infty}.$$

Also, in the same way as we proved Lemma 12, by using **(Φ2)**–**(Φ4)**, we can show that

Lemma 17. *For any integers k, l and m with $k \leq l$, we have*

- (i) $m \geq 0, k+l-m \geq 0 \Rightarrow C_n - D_{n+k-l} = 0 \quad (n \geq l).$
- (ii) $m \geq 0, k+l-m < 0 \Rightarrow \tilde{C}_n - \tilde{D}_{n+k-l} = 0 \quad (n \geq m-k).$
- (iii) $m < 0, k+l-m \geq 0 \Rightarrow C_n - D_{n+k-l} = 0 \quad (n \geq l-m).$
- (iv) $m < 0, k+l-m < 0 \Rightarrow \tilde{C}_n - \tilde{D}_{n+k-l} = 0 \quad (n \geq -k).$

It immediately follows from Lemmas 16 and 17 that

Lemma 18. *For any integers k, l and m with $k \leq l$, the series \tilde{Y} can be expressed as*

$$(2.21) \quad \tilde{Y} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \lambda_2 (1-q)^{m-k-l} x^{-\alpha-\beta+1-\max\{m,0\}-\max\{k+l-m,0\}} \\ \times \frac{(q/x)_\infty}{(cq^{1-\max\{k+l-m,0\}}/(abx))_\infty} \tilde{P} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x \right),$$

where λ_2 is defined as in Lemma 16.

From (2.14), (2.15) and Lemma 18, we obtain the following lemma which gives expressions for \tilde{Q} and \tilde{R} as rational functions.

Lemma 19. *Assume that $k \leq l$. Then, the coefficients of (2.1) can be expressed as*

$$\begin{aligned} \tilde{Q}(k, l, m) &= (-1)^{k+l-m+M} (1-q)^{m-k-l+1} q^{\{k(k-1)+l(l-1)-m(m-1)+M(M-1)\}/2} \\ &\quad \times \frac{a^k b^l}{(b-a)c^{m-1}} \left(\frac{ab}{c} \right)^M \frac{x^{1-\max\{m,0\}}}{(abqx/c)_{M-1}} \tilde{P} \left(\begin{matrix} k, l \\ m \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; q, x \right), \\ \tilde{R}(k, l, m) &= (-1)^{k+l-m-1+M'} (1-q)^{m-k-l} q^{\{k(k-1)+l(l-1)-m(m-1)+M'(M'+1)\}/2} \\ &\quad \times \frac{(1-a)(1-b)a^k b^l}{(b-a)c^m} \left(\frac{ab}{c} \right)^{M'-1} \frac{x^{-\max\{m-1,0\}}}{(abqx/c)_{M'}} \tilde{P} \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x \right), \end{aligned}$$

where $M := \max\{k+l-m, 0\}$ and $M' := \max\{k+l-m-1, 0\}$.

Proof. By substituting $(k, l, m) = (1, 1, 1)$ into (2.21), we have

$$(2.22) \quad \tilde{Y} \left(\begin{matrix} 1, 1 \\ 1 \end{matrix}; \begin{matrix} a, b \\ c \end{matrix}; x \right) = \lambda_2 \frac{x^{-\alpha-\beta-1}}{1-q} \frac{(q/x)_\infty}{(c/(abx))_\infty} \frac{(b-a)c}{a^2 b^2}.$$

From (2.14), (2.21) and (2.22), we have the expression for \tilde{Q} .

Replacing k, l, m, a, b, c in (2.21) with respective $k-1, l-1, m-1, aq, bq, cq$ yields

$$(2.23) \quad \tilde{Y} \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; x \right) \\ = \lambda_2 (1-q)^{m-k-l-1} x^{-\alpha-\beta-1-\max\{m-1,0\}-\max\{k+l-m-1,0\}} \\ \times \frac{(q/x)_\infty}{(cq^{-\max\{k+l-m-1,0\}}/(abx))_\infty} \tilde{P} \left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x \right).$$

From (2.15), (2.22) and (2.23), we have the expression for \tilde{R} . Thus the lemma is proved. \square

From (2.2), (2.3) and Lemma 19, we can complete the proof of Theorem 3.

2.6. Proofs of Corollaries 4 and 6.

By using Theorems 1–3, we show that Corollaries 4 and 6.

First, we prove Corollary 4. From Theorem 1, we find that Q in Theorem 2 and Q in Theorem 3 are equal to each other. Therefore, by comparing the expressions for Q in Theorems 2 and 3, we obtain (1.3). Also, we can easily verify that the degree of P equals the degree of \tilde{P} from (1.3). Let $d := \max\{m, 0\} + \max\{k + l - m, 0\} - k - 1$. Then, we have

$$d = \begin{cases} l - 1 & (m \geq 0, k + l - m \geq 0), \\ m - k - 1 & (m \geq 0, k + l - m < 0), \\ l - m - 1 & (m < 0, k + l - m \geq 0), \\ -k - 1 & (m < 0, k + l - m < 0). \end{cases}$$

It follows from Definition 15 that for any integers k , l and m with $k \leq l$, the degree of the polynomial \tilde{P} is no more than d , and the coefficient of x^d in \tilde{P} equals $(C_0 - D_0)|_{k=l}$ when $k = l$, and equals C_0 when $k < l$; Namely, the coefficient of x^d in \tilde{P} equals

$$\begin{cases} \left(\frac{c}{ab}\right)^k q^{k(m-k-l+1)} \left\{ a^{m-2k} (a)_k (aq/c)_{k-m} - b^{m-2k} (b)_k (bq/c)_{k-m} \right\} & (k = l), \\ \left(\frac{c}{ab}\right)^k q^{k(m-k-l+1)} a^{m-k-l} \frac{(a)_k (aq/c)_{k-m}}{(aq/b)_{k-l}} & (k < l). \end{cases}$$

This completes the proof of Corollary 4.

Next, we prove Corollary 6. From Theorem 2, we have

$$\begin{aligned} & Q(k-1, l-1, m-1)|_{(a,b,c) \mapsto (aq,bq,cq)} \\ &= -\frac{(1-aq)(1-bq)c}{(1-c)(1-cq)} \frac{x^{1-\max\{m-1,0\}}}{(abq^2x/c)_{\max\{k+l-m-1,0\}-1}} P\left(\begin{matrix} k-1, l-1 \\ m-1 \end{matrix}; \begin{matrix} aq, bq \\ cq \end{matrix}; q, x\right) \\ &= \frac{(1-aq)(1-bq)x(c-abqx)}{(1-c)(1-cq)} R(k, l, m). \end{aligned}$$

Thus Corollary 6 is proved. We can also obtain Corollary 6 from Theorem 3 in the same way.

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